Matrix Algebra and R

1 Matrices

A matrix is a two dimensional array of numbers. The number of rows and number of columns defines the order of the matrix. Matrices are denoted by boldface capital letters.

1.1 Examples

$$\mathbf{A} = \begin{pmatrix} 7 & 18 & -2 & 22 \\ -16 & 3 & 55 & 1 \\ 9 & -4 & 0 & 31 \end{pmatrix}_{\mathbf{3\times 4}}$$

$$\mathbf{B} = \begin{pmatrix} x & y+1 & x+y+z \\ a-b & c\log d & e \\ \sqrt{x-y} & (m+n)/n & p \end{pmatrix}_{\mathbf{3\times 3}}$$

and

$$\mathbf{C} = \begin{pmatrix} \mathbf{C_{11}} & \mathbf{C_{12}} \\ \mathbf{C_{21}} & \mathbf{C_{22}} \end{pmatrix}_{\mathbf{2\times 2}}$$

1.2 Making a Matrix in R

```
A = matrix(data=c(7,18,-2,22,-16,3,55,1,9,-4,0,31),byrow=TRUE,
nrow=3,ncol=4)

# Check the dimensions
dim(A)
```

1.3 Vectors

Vectors are matrices with either one row (row vector) or one column (column vector), and are denoted by boldface small letters.

1.4 Scalar

A scalar is a matrix with just one row and one column, and is denoted by an letter or symbol.

2 Special Matrices

2.1 Square Matrix

A matrix with the same number of rows and columns.

2.2 Diagonal Matrix

Let $\{a_{ij}\}$ represent a single element in the matrix $\mathbf{A}$. A diagonal matrix is a square matrix in which all a_{ij} are equal to zero except when i equals j.

2.3 Identity Matrix

This is a diagonal matrix with all a_{ii} equal to one (1). An identity matrix is usually written as $\mathbf{I}$.

To make an identity matrix with r rows and columns, use

```
id = function(n) diag(c(1),nrow=n,ncol=n)

# To create an identity matrix of order 12
I12 = id(12)
```

2.4 J Matrix

A $\mathbf{J}$ matrix is a general matrix of any number of rows and columns, but in which all elements in the matrix are equal to one (1).

The following function will make a $\mathbf{J}$ matrix, given the number or rows, r, and number of columns, c.

```
jd = function(n,m) matrix(c(1),nrow=n,ncol=m)

# To make a matrix of 6 rows and 10 columns of all ones
M = jd(6,10)
```

2.5 Null Matrix

A null matrix is a **J** matrix multiplied by 0. That is, all elements of a null matrix are equal to 0.

2.6 Triangular Matrix

A lower triangular matrix is a square matrix where elements with j greater than i are equal to zero (0), $\{ a_{ij}\}$ equal 0 for j greater than i. There is also an upper triangular matrix in which $\{a_{ij}\}$ equal 0 for i greater than j.

2.7 Tridiagonal Matrix

A tridiagonal matrix is a square matrix with all elements equal to zero except the diagonals and the elements immediately to the left and right of the diagonal. An example is shown below.

$$\mathbf{B} = \begin{pmatrix} 10 & 3 & 0 & 0 & 0 & 0 \\ 3 & 10 & 3 & 0 & 0 & 0 \\ 0 & 3 & 10 & 3 & 0 & 0 \\ 0 & 0 & 3 & 10 & 3 & 0 \\ 0 & 0 & 0 & 3 & 10 & 3 \\ 0 & 0 & 0 & 0 & 3 & 10 \end{pmatrix}.$$

3 Matrix Operations

3.1 Transposition

Let $\{a_{ij}\}$ represent a single element in the matrix $\mathbf{A}$. The transpose of $\mathbf{A}$ is defined as

$$\mathbf{A}' = \{a_{ji}\}.$$

If $\mathbf{A}$ has r rows and c columns, then $\mathbf{A}'$ has c rows and r columns.

$$\mathbf{A} = \begin{pmatrix} 7 & 18 & -2 & 22 \\ -16 & 3 & 55 & 1 \\ 9 & -4 & 0 & 31 \end{pmatrix}$$

$$\mathbf{A}' = \begin{pmatrix} 7 & -16 & 9 \\ 18 & 3 & -4 \\ -2 & 55 & 0 \\ 22 & 1 & 31 \end{pmatrix}.$$

In R,

```
At = t(A)
# t() is the transpose function
```

3.2 Diagonals

The diagonals of matrix $\mathbf{A}$ are $\{a_{ii}\}$ for i going from 1 to the number of rows in the matrix.

Off-diagonal elements of a matrix are all other elements excluding the diagonals.

Diagonals can be extracted from a matrix in R by using the `diag()` function.

3.3 Addition of Matrices

Matrices are *conformable for addition* if they have the same order. The resulting sum is a matrix having the same number of rows and columns as the two matrices to be added. Matrices that are not of the same order cannot be added together.

$$\mathbf{A} = \{\mathbf{a_{ij}}\} \text{ and } \mathbf{B} = \{\mathbf{b_{ij}}\}$$

$$\mathbf{A} + \mathbf{B} = \{\mathbf{a_{ij}} + \mathbf{b_{ij}}\}.$$

An example is

$$\mathbf{A} = \begin{pmatrix} 4 & 5 & 3 \\ 6 & 0 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 1 \end{pmatrix}$$

then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 4+1 & 5+0 & 3+2 \\ 6+3 & 0+4 & 2+1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 5 & 5 \\ 9 & 4 & 3 \end{pmatrix} = \mathbf{B} + \mathbf{A}.$$

Subtraction is the addition of two matrices, one of which has all elements multiplied by a minus one (-1). That is,

$$\mathbf{A} + (-1)\mathbf{B} = \begin{pmatrix} 3 & 5 & 1 \\ 3 & -4 & 1 \end{pmatrix}.$$

R will check matrices for conformability, and will not perform the operation unless they are conformable.

3.4 Multiplication of Matrices

Two matrices are *conformable for multiplication* if the number of columns in the first matrix equals the number of rows in the second matrix.

If $\mathbf{C}$ has order $p \times q$ and $\mathbf{D}$ has order $m \times n$, then the product $\mathbf{CD}$ exists only if $q = m$. The product matrix has order $p \times n$.

In general, $\mathbf{CD}$ does not equal $\mathbf{DC}$, and most often the product $\mathbf{DC}$ may not even exist because $\mathbf{D}$ may not be conformable for multiplication with $\mathbf{C}$. Thus, the ordering of matrices in a product must be carefully and precisely written.

The computation of a product is defined as follows: let

$$\mathbf{C}_{p\times q} = \{c_{ij}\}$$

and

$$\mathbf{D}_{m\times n} = \{d_{ij}\}$$

and $q = m$, then

$$\mathbf{CD}_{p\times n} = \{\sum_{k=1}^{m} c_{ik}d_{kj}\}.$$

As an example, let

$$\mathbf{C} = \begin{pmatrix} 6 & 4 & -3 \\ 3 & 9 & -7 \\ 8 & 5 & -2 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{pmatrix},$$

then

$$\mathbf{CD} = \begin{pmatrix} 6(1)+4(2)-3(3) & 6(1)+4(0)-3(-1) \\ 3(1)+9(2)-7(3) & 3(1)+9(0)-7(-1) \\ 8(1)+5(2)-2(3) & 8(1)+5(0)-2(-1) \end{pmatrix} = \begin{pmatrix} 5 & 9 \\ 0 & 10 \\ 12 & 10 \end{pmatrix}.$$

In R,

```
# C times D - conformability is checked
CD = C %*% D
```

3.5 Traces of Square Matrices

The trace is the sum of the diagonal elements of a matrix. The sum is a scalar quantity. Let

$$\mathbf{A} = \begin{pmatrix} .51 & -.32 & -.19 \\ -.28 & .46 & -.14 \\ -.21 & -.16 & .33 \end{pmatrix},$$

then the trace is

$$tr(\mathbf{A}) = .51 + .46 + .33 = 1.30.$$

In R, the trace is achieved using the `sum()` and `diag()` functions together. The `diag()` function extracts the diagonals of the matrix, and the `sum()` function adds them together.

```
# Trace of the matrix A
trA = sum(diag(A))
```

3.6 Direct Sum of Matrices

For matrices of any dimension, say $\mathbf{H}_1$, $\mathbf{H}_2$, $\ldots \mathbf{H}_n$, the direct sum is

$$\sum_i^+ \mathbf{H}_i = \mathbf{H}_1 \oplus \mathbf{H}_2 \oplus \cdots \oplus \mathbf{H}_n = \begin{pmatrix} \mathbf{H}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}_n \end{pmatrix}.$$

In R, the direct sum is accomplished by the `block()` function which is shown below.

```
# Direct sum operation via the block function

block <- function( ...  ) {
argv = list( ...  )
i = 0
for( a in argv ) {
m = as.matrix(a)
if(i == 0)
rmat = m
else
{
nr = dim(m)[1]
nc = dim(m)[2]
aa = cbind(matrix(0,nr,dim(rmat)[2]),m)
rmat = cbind(rmat,matrix(0,dim(rmat)[1],nc))
rmat = rbind(rmat,aa)
}
i = i+1
}
rmat
}
```

To use the function,

```
Htotal = block(H1,H2,H3,H4)
```

3.7 Kronecker Product

The Kronecker product, also known as the direct product, is where every element of the first matrix is multiplied, as a scalar, times the second matrix. Suppose that $\mathbf{B}$ is a matrix of order $m \times n$ and that $\mathbf{A}$ is of order 2×2, then the direct product of $\mathbf{A}$ times $\mathbf{B}$ is

$$\mathbf{A} \otimes \mathbf{B} \;=\; \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} \end{pmatrix}.$$

Notice that the dimension of the example product is $2m \times 2n$.

In R, a direct product can be obtained as follows:

```
AB = A %x% B
# Note the small x between % %
```

4 Matrix Inversion

An inverse of a square matrix $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$. An inverse of a matrix pre- or post-multiplied times the original matrix yields an identity matrix. That is,

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}, \text{ and } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

A matrix can be inverted if it has a nonzero determinant.

4.1 Determinant of a Matrix

The determinant of a matrix is a single scalar quantity. For a 2×2 matrix, say

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then the determinant is

$$|\mathbf{A}| = a_{11}a_{22} - a_{21}a_{12}.$$

For a 3×3 matrix, the determinant can be reduced to a series of determinants of 2×2 matrices. For example, let

$$\mathbf{B} = \begin{pmatrix} 6 & -1 & 2 \\ 3 & 4 & -5 \\ 1 & 0 & -2 \end{pmatrix},$$

then

$$\begin{aligned} |\mathbf{B}| &= 6\begin{vmatrix} 4 & -5 \\ 0 & -2 \end{vmatrix} - 1(-1)\begin{vmatrix} 3 & -5 \\ 1 & -2 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} \\ &= 6(-8) \;+ 1(-1) \;+ 2(-4) \end{aligned}$$

$$= -57.$$

In R, the `det()` function may be used to compute the determinant.

In R, there are different ways to compute an inverse.

```
BI = ginv(B) # will give generalized inverse if
# determinant is zero
```

4.2 Inverse of an Inverse

The inverse of an inverse matrix is equal to the original matrix. That is,

$$(\mathbf{A^{-1}})^{-1} = \mathbf{A}.$$